Mixed Symmetry Solutions of Generalized Three-Particle Bargmann-Wigner Equations in the Strong-Coupling Limit

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Abstract

Starting from a nonlinear isospinor-spinor field equation, generalized three-particle Bargmann-Wigner equations are derived. In the strong-coupling limit, a special class of spin 1/2 bound-states are calculated. These solutions which are antisymmetric with respect to all indices, have mixed symmetries in isospin-superspin space and in spin orbit space. As a consequence of this mixed symmetry, we get three solution manifolds. In appendix B, table 2, these solution manifolds are interpreted as the three generations of leptons and quarks. This interpretation will be justified in a forthcoming paper.

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1 Introduction

In various field theoretic models, three-fermion bound states are assumed to play an important role. In general, ordinary Schrödinger equations or Bethe Salpeter equations are used for their calculation. As far as these models are based on nonlinear spinor equations, for instance Nambu Lasinio Models or Heisenberg Models, it is reasonable to apply generalized Bargmann Wigner equations for the calculation of many-fermion bound states. In previous papers we calculated two-fermion composites and three-fermion composites by means of generalized B. W. equations [3],[1], [5]. In [2],[3], [8] we showed that the effective dynamics of these bound states leads to an unbroken SU(2)×U(1) gauge theory and that the three-fermion bound states may be interpreted as quarks and leptons. However, because we used three-particle states with symmetric isospin-superspin dependence, we obtained isospin quartets instead of isospin doublets which are required by phenomenology. Furthermore, the

three-particle solutions with symmetric isospin-superspin dependence can only describe one generation of leptons and quarks i. e. the three families have not been included up to now.

This paper is a first step in order to remove these drawbacks. Starting from a nonlinear spinor-isospinor field equation we will derive generalized Bargmann-Wigner equations. Then we will calculate those solutions which have mixed symmetry in isospin-superspin space and spin orbit space. The calculations are performed in the strong-coupling limit. A physical motivation for the application of the strong-coupling limit is given in [5], where also literature concerning the strong-coupling limit is cited. In the mixed symmetric sector we not only get isospin doublets but also three solution manifolds which can in principle describe the three generations of leptons and quarks. The proof that the effective dynamics of mixed symmetric three-fermion states and two-fermion composites includes the three generations of leptons and quarks is postponed to a forthcoming paper.

It should be mentioned that we can choose between the solution manifold with symmetric isospin-superspin dependence and the solution manifold with mixed symmetry in isospin-superspin space by fine-tuning the coupling constant. The value of the coupling constant can be chosen such that one of the two solution manifolds acquire low masses whereas the masses of the remaining solution manifold get high values i. e. it become unobservable.

The paper is organized as follows. In section 2 we introduce the subfermion model. In section 3, the three-particle equations are given and the strong-coupling limit is performed. In section 4 the solutions of these three-particle equations are discussed for the case of mixed symmetry in isospin-superspin space. In section 5 a summary is given. Finally we mention that some definitions and notations used in this paper are given in appendix A and B.

2 The model

The basic fermions of our model are described by Dirac spinors which satisfy the following nonlinear spinor-isospinor equation:

$$(i\gamma^\mu \partial_\mu - m)_{\alpha\beta}\, \delta_{AB}\psi_{\beta B}(x) = gV_{\alpha\beta\gamma\delta}^{ABCD}\psi_{\beta B}(x)\bar{\psi}_{\gamma C}(x)\psi_{\delta D}(x) \tag{1}$$

with

$$V_{\alpha\beta\gamma\delta}^{ABCD} = \frac{1}{2}\sum_{h=1}^{2}\left(v_{\alpha\beta}^{h}\delta_{AB}v_{\gamma\delta}^{h}\delta_{CD} - v_{\alpha\delta}^{h}\delta_{AD}v_{\gamma\beta}^{h}\delta_{CB}\right)$$

$$v_{\alpha\beta}^{1} := \delta_{\alpha\beta}\,,\; v_{\alpha\beta}^{2} := i\gamma_{\alpha\beta}^{5} \quad .$$

As leptons and quarks are assumed to be constituted by three of these fermions we call them in the following subfermions, in contrast to the fermions of the standard model.

If we use the charge conjugated spinor $\varphi_{\alpha A}^c$ instead of the adjoint spinor $\bar{\varphi}_{\alpha A}$ and furthermore introduce the definitions

$$
\begin{aligned}
\phi_{\alpha\kappa} &:= \{\psi_{\alpha 1}\,,\; \psi_{\alpha 2}\,,\; \psi_{\alpha 1}^c\,,\; \psi_{\alpha 2}^c\} \\
Z &:= (\alpha, \kappa) \\
D_{Z_1 Z_2}^{\mu} &:= i\gamma_{\alpha_1 \alpha_2}^{\mu} \delta_{\kappa_1 \kappa_2} \\
m_{Z_1 Z_2} &:= m\delta_{\alpha_1 \alpha_2} \delta_{\kappa_1 \kappa_2} \\
U_{Z_1 Z_2 Z_3 Z_4}^{h} &:= g v_{\alpha_1 \alpha_2}^h \left(v^h C \right)_{\alpha_3 \alpha_4} \delta_{\kappa_1 \kappa_2} \gamma_{\kappa_3 \kappa_4}^5 \quad,
\end{aligned}
$$

we can combine (1) and its charge conjugated equation into one equation

$$
\left(D_{Z_1 Z_2}^{\mu} \partial_\mu - m_{Z_1 Z_2} \right) \phi_{Z_2}(\mathbf{r}, t) = \sum_h U_{Z_1 \{Z_2 Z_3 Z_4\}_{as}}^{h} \phi_{Z_2}(\mathbf{r}, t)\phi_{Z_3}(\mathbf{r}, t)\phi_{Z_4}(\mathbf{r}, t) \quad (2)
$$

The canonical equal time anticommutator then reads

$$
\begin{aligned}
\{\phi_{Z_1}(\mathbf{r}_1, t), \phi_{Z_2}(\mathbf{r}_2, t)\} &= A_{Z_1 Z_2}\delta(\mathbf{r}_1 - \mathbf{r}_2) \\
A_{Z_1 Z_2} &:= \gamma_{\kappa_1 \kappa_2}^5 (C\gamma^0)_{\alpha_1 \alpha_2} \quad.
\end{aligned}
$$

We characterize the quantum states $|a\rangle$ of the model (2) by the set of normal ordered matrix-elements for equal times t

$$
\varphi_n\left(\mathbf{r}_1, Z_1, \ldots, \mathbf{r}_n, Z_n \middle| a\right) := \langle 0 \left| \mathcal{N}\{\phi_{Z_1}(\mathbf{r}_1, t)\ldots\phi_{Z_n}(\mathbf{r}_n, t)\}\right| a\rangle \quad. \quad (3)
$$

Introducing furthermore the generating functional states for the normal transformed matrix-elements $|\mathcal{F}(j, a)\rangle$ with anticommuting sources $j_Z(\mathbf{r})$ and their corresponding duals $\partial_Z(\mathbf{r})$, we get as a compact formulation of the field dynamics the functional energy equation

$$
\begin{aligned}
(E_a - E_0)|\mathcal{F}(j, a)\rangle = &\; K_{I_1 I_2} j_{I_1} \partial_{I_2}|\mathcal{F}(j, a)\rangle \\
&+ \sum_h W_{I_1 I_2 I_3 I_4}^h \Big\{ j_{I_1} \partial_{I_4} \partial_{I_3} \partial_{I_2} - 3F_{I_4 I}^a j_{I_1} j_I \partial_{I_3} \partial_{I_2} \\
&+ \left(3F_{I_4 I}^a F_{I_3 I'}^a + \frac{1}{4} A_{I_4 I} A_{I_3 I'} \right) j_{I_1} j_I j_{I'} \partial_{I_2} \\
&- \left(F_{I_4 I}^a F_{I_3 I'}^a + \frac{1}{4} A_{I_4 I} A_{I_3 I'} \right) F_{I_2 I''}^a j_{I_1} j_I j_{I'} j_{I''} \Big\} |\mathcal{F}(j, a)\rangle.
\end{aligned} \quad (4)
$$

In (4) we used the definitions

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$$K_{I_1 I_2} := i D^0_{Z_1 Z} \left(\vec{D}_{ZZ_2} \cdot \nabla_{\mathbf{r}_1} - m_{ZZ_2} \right) \delta \left(\mathbf{r}_1 - \mathbf{r}_2 \right)$$

$$W^h_{I_1 I_2 I_3 I_4} := i D^0_{Z_1 Z} U^h_{Z\{Z_2 Z_3 Z_4\}_{as}} \delta \left(\mathbf{r}_1 - \mathbf{r}_2 \right) \delta \left(\mathbf{r}_1 - \mathbf{r}_3 \right) \delta \left(\mathbf{r}_1 - \mathbf{r}_4 \right)$$

$$A_{I_1 I_2} := A_{Z_1 Z_2} \delta \left(\mathbf{r}_1 - \mathbf{r}_2 \right)$$

$$|\mathcal{F}(j,a)\rangle := \sum_{n=0}^{\infty} \frac{i^n}{n!} \langle 0 \left| N \{\phi_{I_1} \dots \phi_{I_n}\} \right| a \rangle \, j_{I_1} \dots j_{I_n} |0\rangle_f \quad ,$$

where the functional state $|0\rangle_f$ satisfies $\partial_I |0\rangle_f = {}_f\langle 0| j_I = 0$ and F^a in (4) is the antisymmetric equal time limit of the fermion field propagator. Furthermore it should be emphasized, that at this stage of calculation we have not restricted ourselves onto Fock-space.

3 Three-particle equations

If projected in coordinate space, (4) yields an infinite set of coupled differential equations for the infinite set of matrix-elements of normal ordered products of field operators. In order to obtain generalized Bargmann-Wigner equations from this set, we consider only the "diagonal part" of (4) which is given by

$$\omega |\mathcal{F}(j,a)\rangle^d = j_{I_1} K_{I_1 I_2} \partial_{I_2} |\mathcal{F}(j,a)\rangle^d - 3 \sum_{h=1}^{2} j_{I_1} W^h_{I_1 I_2 I_3 I_4} F^a_{I_4 I'} j_{I'} \partial_{I_3} \partial_{I_2} |\mathcal{F}(j,a)\rangle^d . \quad (5)$$

The non-diagonal part of (4) is assumed to mediate the interactions of the eigenstates of (5) [2]. The corresponding theory of effective interactions is not the topic of this paper. Rather we want to investigate the solutions of (5), in particular three-particle solutions. By projecting with $(\frac{1}{i})^3 {}_f\langle 0| \partial_{V_3} \partial_{V_2} \partial_{V_1}$ from the left we obtain from (5) the set of equations

$$\omega \, \varphi_{V_1 V_2 V_3} = K_{V_1 I} \, \varphi_{I V_2 V_3} + K_{V_2 I} \, \varphi_{V_1 I V_3} + K_{V_3 I} \, \varphi_{V_1 V_2 I}$$

$$-3 \sum_{p \in S(3)} (-)^p \sum_{h=1}^{2} W^h_{V_{p(1)} I_1 I_2 I_3} F^a_{I_3 V_{p(2)}} \, \varphi_{I_1 I_2 V_{p(3)}} \quad (6)$$

for the calculation of the three-particle amplitude $\varphi_{V_1 V_2 V_3}$ which has to be antisymmetric with respect to all indices. For the connection of (6) with ordinary BW-equations we refer to [5] and the literature cited therein.

For a first draft, we consider eqn. (6) in the strong-coupling limit which is characterized by

$$K_{II'} \rightarrow -im D^0_{II'} \quad , \quad -im D^0_{II'} = m \delta_{\kappa\kappa'} \gamma^0_{\alpha\alpha'} \delta(\mathbf{r} - \mathbf{r}') \quad . \quad (7)$$

For a physical motivation with respect to the use of the strong-coupling limit see [5]. With (7), eqn. (6) becomes

$$\omega\,\varphi_{V_1V_2V_3} + im\left(D^0_{V_1I}\,\varphi_{IV_2V_3} + D^0_{V_2I}\,\varphi_{V_1IV_3} + D^0_{V_3I}\,\varphi_{V_1V_2I}\right)$$

$$= -3\sum_{p\in S(3)}(-)^p\sum_{h=1}^{2}W^h_{V_{p(1)}I_1I_2I_3}F^a_{I_3V_{p(2)}}\,\varphi_{I_1I_2V_{p(3)}} \quad . \tag{8}$$

For the evaluation of (8) we need the explicit form of F^a. As a first approximation we take for F^a the antisymmetric equal time limit of the *free* fermion field propagator, which we assume to be regularized in the sense that $F^a|_{\mathbf{r}_1=\mathbf{r}_2} < \infty$ (For a systematic treatment of nonperturbative regularization see [4]). Therefore we have

$$F^a\begin{pmatrix}\mathbf{r}\\ \alpha_1\ \alpha_2\\ \kappa_1\ \kappa_2\end{pmatrix} = (\gamma_5)_{\kappa_1\kappa_2}\,[i\vec{\gamma}C\cdot\nabla_{\mathbf{r}} + mC]_{\alpha_1\alpha_2}\,\mathrm{Reg}\,\frac{-1}{2(2\pi)^3}\int d\mathbf{p}\,\frac{e^{-i\mathbf{p}\mathbf{r}}}{\sqrt{m^2+\mathbf{p}^2}}$$

$$=: (\gamma_5)_{\kappa_1\kappa_2}F\begin{pmatrix}\mathbf{r}\\ \alpha_1\ \alpha_2\end{pmatrix}$$

$$=: (\gamma_5)_{\kappa_1\kappa_2}\left(\gamma^k C\,h^k(\mathbf{r}) + C\,s(\mathbf{r})\right)_{\alpha_1\alpha_2} \tag{9}$$

where $\mathbf{r} := \mathbf{r}_1 - \mathbf{r}_2$, $h^k(-\mathbf{r}) = -h^k(\mathbf{r})$. Furthermore we mention that $s(\mathbf{r})$ is a scalar function which means that it is a function of $\mathbf{r}^2 := \mathbf{r}\cdot\mathbf{r}$. Using Fierz identities, the vertex $\sum_{h=1}^{2}W^h_{II_1I_2I_3}$ can be written as

$$3\sum_{h=1}^{2}W^h_{II_1I_2I_3} = g\,\delta(\mathbf{r}-\mathbf{r}_1)\,\delta(\mathbf{r}-\mathbf{r}_2)\,\delta(\mathbf{r}-\mathbf{r}_3)\cdot$$

$$\left\{4(\gamma^0\gamma^\mu)_{\alpha\alpha_3}\frac{1}{4}(\gamma^\mu C)^\dagger_{\alpha_1\alpha_2}(A^n\gamma_5)_{\kappa\kappa_3}\hat{A}^n_{\kappa_1\kappa_2} + 4(\gamma^0\gamma^\mu\gamma_5)_{\alpha\alpha_3}\frac{1}{4}(\gamma^\mu\gamma_5 C)^\dagger_{\alpha_2\alpha_1}(S^m\gamma_5)_{\kappa\kappa_3}\hat{S}^m_{\kappa_1\kappa_2}\right.$$

$$\left. -4\gamma^0_{\alpha\alpha_3}\frac{1}{4}C^\dagger_{\alpha_2\alpha_1}(\gamma_5\cdot\gamma_5)_{\kappa\kappa_3}(\gamma_5)_{\kappa_1\kappa_2} + 4(\gamma^0\gamma_5)_{\alpha\alpha_3}\frac{1}{4}(\gamma_5 C)^\dagger_{\alpha_2\alpha_1}(\gamma_5\cdot\gamma_5)_{\kappa\kappa_3}(\gamma_5)_{\kappa_1\kappa_2}\right\} \tag{10}$$

where A^n or S^m is an arbitrary complete set of antisymmetric or symmetric 4×4 matrices, and $\hat{A}^n$ or $\hat{S}^m$ resp. are the corresponding duals which satisfy the following completeness and orthogonality relations:

$$-\mathrm{tr}\left[\hat{A}^n A^{n'}\right] = \delta_{nn'} \quad , \quad A^n_{\kappa_1\kappa_2}\hat{A}^n_{\kappa'_1\kappa'_2} = \frac{1}{2}\left(\delta_{\kappa_1\kappa'_1}\delta_{\kappa_2\kappa'_2} - \delta_{\kappa_1\kappa'_2}\delta_{\kappa_2\kappa'_1}\right)$$

$$\mathrm{tr}\left[\hat{S}^m S^{m'}\right] = \delta_{mm'} \quad , \quad S^m_{\kappa_1\kappa_2}\hat{S}^m_{\kappa'_1\kappa'_2} = \frac{1}{2}\left(\delta_{\kappa_1\kappa'_1}\delta_{\kappa_2\kappa'_2} + \delta_{\kappa_1\kappa'_2}\delta_{\kappa_2\kappa'_1}\right) \quad .$$

4 Calculation of three-particle states in the mixed symmetric sector

In this section, we discuss those spin $1/2$ solutions of (8) which possess mixed symmetry in isospin-superspin space. In addition, we restrict ourselves to solutions φ which satisfy the condition

$$(\gamma_5)_{\kappa_1\kappa_2} \, \varphi \begin{pmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \kappa_1 & \kappa_2 & \kappa_3 \end{pmatrix} = 0 \tag{11}$$

As will be shown below, the constraint (11) enables us to completely separate the determination of the isospin-superspin part from the calculation of the spin orbit part of the wave function. This is in full analogy to the case of symmetric isospin-superspin dependence (see [5]).

The ansatz for an antisymmetric function $\varphi_{I_1 I_2 I_3}$ with mixed symmetry in isospin-superspin space reads [6]:

$$\begin{aligned} \left| \varphi^{j,a} \right\rangle = {} & C_{11} \left| j \right\rangle \otimes C_{22} \left| \Phi^a \right\rangle - C_{21} \left| j \right\rangle \otimes C_{12} \left| \Phi^a \right\rangle \\ & + C_{22} \left| j \right\rangle \otimes C_{11} \left| \Phi^a \right\rangle - C_{12} \left| j \right\rangle \otimes C_{21} \left| \Phi^a \right\rangle \end{aligned} \tag{12}$$

The Young-operators C_{ik} are defined in appendix A. The quantum number a represents the $J = 1/2$ spin quantum numbers whereas the quantum number j of the isospin state $\left| j \right\rangle$ combines the isospin and fermion quantum numbers. The possibility to classify the states according to isospin and fermion quantum numbers is a consequence of the global SU(2)×U(1)-invariance of equation (1) or (8) respectively. Furthermore we have used the Dirac bracket formulation. For instance

$$\left\{ \left\langle \kappa_1\kappa_2\kappa_3 \right| \otimes \left\langle \begin{matrix} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{matrix} \right| \right\} \left| \varphi \right\rangle = \varphi \begin{pmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \kappa_1 & \kappa_2 & \kappa_3 \end{pmatrix}$$

$$\left\{ \left\langle \kappa_1\kappa_2\kappa_3 \right| \otimes \left\langle \begin{matrix} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{matrix} \right| \right\} \left\{ \left| j \right\rangle \otimes \left| \Phi \right\rangle \right\} = \Theta^j_{\kappa_1\kappa_2\kappa_3} \, \Phi \begin{pmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix}$$

with $\Theta^j_{\kappa_1\kappa_2\kappa_3} = \left\langle \kappa_1\kappa_2\kappa_3 \, | \, j \right\rangle$, etc.

In general, we must take into account the possibility of degeneracy. Therefore we have to replace in (12) the states $\left| j \right\rangle$ or $\left| \Phi^a \right\rangle$ resp. by the linear combinations $a_s \left| j, s \right\rangle$ or $b_r \left| \Phi^a_r \right\rangle$ respectively, where the degeneracy indices s, r enumerate the states belonging to the same quantum number. With these replacements we get from (12):

$$\begin{aligned} \left| \varphi \right\rangle = a_s \, b_r \Big\{ & C_{11} \left| j, s \right\rangle \otimes C_{22} \left| \Phi^a_r \right\rangle - C_{21} \left| j, s \right\rangle \otimes C_{12} \left| \Phi^a_r \right\rangle \\ & + C_{22} \left| j, s \right\rangle \otimes C_{11} \left| \Phi^a_r \right\rangle - C_{12} \left| j, s \right\rangle \otimes C_{21} \left| \Phi^a_r \right\rangle \Big\} \end{aligned} \tag{13}$$

In the mixed symmetric state space, the unit operator is given by the sum of the projection operators C_{11} and C_{22}. This yields a decomposition of the mixed symmetric state space according to $H_{\text{mixed}} = H_{11} \oplus H_{22}$. If we require the states $C_{11} |j, s\rangle$ to be complete, i. e. every state of H_{11} with quantum number j can uniquely be written as $\alpha_s C_{11} |j, s\rangle$, then it can be proven that we get a complete set of states in H_{22} with the help of the step operator C_{21}:

$$C_{22} |j, s\rangle = \alpha_{ss'} \, C_{21} C_{11} |j, s'\rangle \overset{(39)}{=} \alpha_{ss'} \, C_{21} |j, s'\rangle \tag{14}$$

With the analog requirement for the state $C_{22} |\Phi_r^a\rangle$, we have

$$C_{11} |\Phi_r^a\rangle = \beta_{rr'} \, C_{12} C_{22} |\Phi_{r'}^a\rangle = \beta_{rr'} \, C_{12} |\Phi_{r'}^a\rangle \tag{15}$$

If we substitute (14) and (15) into (13), we get

$$\left| \varphi^{j,a} \right\rangle = a_s b_r \Big\{ C_{11} |j, s\rangle \otimes C_{22} |\Phi_r^a\rangle - C_{21} |j, s\rangle \otimes C_{12} |\Phi_r^a\rangle$$
$$+ \alpha_{ss'} \beta_{rr'} \, C_{21} |j, s'\rangle \otimes C_{12} |\Phi_{r'}^a\rangle - \alpha_{ss'} \beta_{rr'} \, C_{11} |j, s'\rangle \otimes C_{22} |\Phi_{r'}^a\rangle \Big\}$$

which can be written in the form

$$\begin{aligned}
\left| \varphi^{j,a} \right\rangle &= (a_s b_r - a_{s'} \alpha_{s's} b_{r'} \beta_{r'r}) \, C_{11} |j, s\rangle \otimes C_{22} |\Phi_r^a\rangle \\
&\quad - (a_s b_r - a_{s'} \alpha_{s's} b_{r'} \beta_{r'r}) \, C_{21} |j, s\rangle \otimes C_{12} |\Phi_r^a\rangle \\
&=: t_{sr} \Big(C_{11} |j, s\rangle \otimes C_{22} |\Phi_r^a\rangle - C_{21} |j, s\rangle \otimes C_{12} |\Phi_r^a\rangle \Big)
\end{aligned} \tag{16}$$

So far we have not achieved any simplification with respect to (13). However, it is shown in appendix B that due to the requirement (11) there is no degeneracy in the isospin-superspin space i. e with (11) we have

$$C_{11} |j, s\rangle \rightarrow C_{11} |j\rangle$$
$$a_s \, , \, \alpha_{ss'} \rightarrow a \, , \, \alpha$$
$$(a \, b_r - a\alpha \, b_{r'} \beta_{r'r}) \rightarrow \eta_r$$

Hence, we infer from (16):

$$\left| \varphi^{j,a} \right\rangle = \eta_r \, C_{11} |j\rangle \otimes C_{22} |\Phi_r^a\rangle - \eta_r \, C_{21} |j\rangle \otimes C_{12} |\Phi_r^a\rangle \tag{17}$$

The ansatz (17) is indeed a simplification in comparison to the ansatz (13) because in this ansatz we only have coefficients η_s with one index instead of coefficients t_{rs} with two indices. Before we use this ansatz to evaluate (8), we discuss a further consequence of (11).

By projecting in (4) with ${}_f\langle 0| \partial_I$, we see that a condition for $\varphi_{I_1 I_2 I_3}$ to describe a genuine three-particle state and not a polarization cloud of one particle, is given by

$$\sum_{h=1}^{2} W_{I I_1 I_2 I_3}^h \varphi_{I_1 I_2 I_3} \overset{!}{=} 0 \tag{18}$$

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In the following we demonstrate that the condition (11) is sufficient for the fulfillment of (18). After substitution of (17) into (18) we get:

$$\eta_r \, (\gamma^0 \gamma^\mu)_{\alpha\alpha_3} \frac{1}{4} (\gamma^\mu C)^\dagger_{\alpha_1\alpha_2} (\gamma_5)_{\kappa_2\kappa_3} (C_{11}\Theta^j)_{\kappa\kappa_2\kappa_3} (C_{22}\Phi_r^a)_{\alpha_1\alpha_2\alpha_3}$$

$$-\eta_r \, (\gamma^0 \gamma^\mu \gamma_5)_{\alpha\alpha_3} \frac{1}{4} (\gamma^\mu \gamma_5 C)^\dagger_{\alpha_1\alpha_2} (\gamma_5)_{\kappa_2\kappa_3} (C_{21}\Theta^j)_{\kappa\kappa_2\kappa_3} (C_{12}\Phi_r^a)_{\alpha_1\alpha_2\alpha_3} \overset{!}{=} 0 \qquad (19)$$

where we already have taken into account that due to (11), the last two terms on the right hand side of (10) do not give any contribution in (18). In order to further simplify (19), we have to separate the isospin-superspin part from the spin part. Therefore we need

Lemma 1

$$(\gamma_5)_{\kappa_2\kappa_3} (C_{11}\Theta^j)_{\kappa_1\kappa_2\kappa_3} = \sqrt{3} (\gamma_5)_{\kappa_2\kappa_3} (C_{21}\Theta^j)_{\kappa_1\kappa_2\kappa_3}$$

Proof:
Due to $(\gamma_5)_{\kappa_1\kappa_2} = (\gamma_5)_{\kappa_2\kappa_1}$ we have

$$(\gamma_5)_{\kappa_2\kappa_3} (C_{11}\Theta^j)_{\kappa_1\kappa_2\kappa_3} = (\gamma_5)_{\kappa_2\kappa_3} (P_{23}C_{11}\Theta^j)_{\kappa_1\kappa_2\kappa_3} \quad ,$$

where the relation $P_{23}C_{11} \overset{(41)}{=} 1/2 \, C_{11} + 1/2 \sqrt{3} \, C_{21}$ completes the proof. $\qquad \square$

With lemma 1, condition (19) is simplified and yields

$$(\gamma_5)_{\kappa_1\kappa_2} (C_{11}\Theta^j)_{\kappa\kappa_1\kappa_2} \cdot \left\{ \eta_r \, (\gamma^0 \gamma^\mu)_{\alpha\alpha_3} \frac{1}{4} (\gamma^\mu C)^\dagger_{\alpha_1\alpha_2} (C_{22}\Phi_r^a)_{\alpha_1\alpha_2\alpha_3} \right.$$

$$\left. -\eta_r \, \frac{1}{\sqrt{3}} \, (\gamma^0 \gamma^\mu \gamma_5)_{\alpha\alpha_3} \frac{1}{4} (\gamma^\mu \gamma_5 C)^\dagger_{\alpha_1\alpha_2} (C_{12}\Phi_r^a)_{\alpha_1\alpha_2\alpha_3} \right\} \overset{!}{=} 0 \qquad (20)$$

We prove that (20) is automatically fulfilled if (11) is postulated. Substituting the ansatz (17) into (11) yields

$$(\gamma_5)_{\kappa_1\kappa_2} (C_{11}\Theta^j)_{\kappa_1\kappa_2\kappa_3} = 0 \qquad\qquad (21)$$
$$(\gamma_5)_{\kappa_1\kappa_2} (C_{21}\Theta^j)_{\kappa_1\kappa_2\kappa_3} = 0 \qquad\qquad (22)$$

Equation (21) is trivially fulfilled due to the antisymmetry of $(C_{11}\Theta^j)$ in the first two indices whereas the requirement (22) is a genuine restriction to the Θ^j. Therefore the requirement (22) is fully equivalent to the requirement (11). In order to get the connection between (22) and (20), we prove the following

Lemma 2

$$(\gamma_5)_{\kappa_2\kappa_3} (C_{21}\Theta^j)_{\kappa_2\kappa_3\kappa_1} = -\frac{2}{\sqrt{3}} (\gamma_5)_{\kappa_2\kappa_3} (C_{11}\Theta^j)_{\kappa_1\kappa_2\kappa_3}$$

Proof:

$$(\gamma_5)_{\kappa_2\kappa_3} (C_{11}\Theta^j)_{\kappa_1\kappa_2\kappa_3} = (\gamma_5)_{\kappa_2\kappa_3} (P_{13}P_{13}C_{11}\Theta^j)_{\kappa_1\kappa_2\kappa_3}$$

$$\overset{(42)}{=} (\gamma_5)_{\kappa_2\kappa_3} \left[P_{13} \left(\frac{1}{2}C_{11} - \frac{\sqrt{3}}{2}C_{21} \right) \Theta^j \right]_{\kappa_1\kappa_2\kappa_3}$$

$$= (\gamma_5)_{\kappa_2\kappa_3} \left[\left(\frac{1}{2}C_{11} - \frac{\sqrt{3}}{2}C_{21} \right) \Theta^j \right]_{\kappa_3\kappa_2\kappa_1}$$

$$= -\frac{\sqrt{3}}{2} (\gamma_5)_{\kappa_2\kappa_3} (C_{21}\Theta^j)_{\kappa_2\kappa_3\kappa_1} \qquad \square$$

Due to lemma 2, we see that (20) is a consequence of the requirement (11) i.e. those functions which fulfill (11) do not describe polarization clouds.

Keeping in mind that due to (11) we can neglect the last two terms on the right hand side of (10), we get after substituting (17) into (8) an eigenvalue equation with the following structure

$$\left\{ \omega - m \left(\gamma_1^0 + \gamma_2^0 + \gamma_3^0 \right) \right\} \left| \varphi^{j,a} \right\rangle = \eta_r \, C^a \circ \left\{ C_{11} \left| j \right\rangle \otimes \hat{O}_1 C_{22} \left| \Phi_r^a \right\rangle \right.$$
$$\left. - C_{21} \left| j \right\rangle \otimes \hat{O}_2 C_{12} \left| \Phi_r^a \right\rangle \right\} \qquad (23)$$

where $\hat{O}_{1,2}$ are well defined operators. The operator C^a is the usual antisymmetrisizer. It's Kronecker decomposition is given by [6],p37,prop. 2. 58:

$$C^a = C^a \times C^s + C^s \times C^a + \frac{1}{2}(C_{11} \times C_{22} - C_{21} \times C_{12} + C_{22} \times C_{11}$$
$$- C_{12} \times C_{21}) \qquad (24)$$

With the help of (24),(37) and (39) we get from (23):

$$\left\{ \omega - m \left(\gamma_1^0 + \gamma_2^0 + \gamma_3^0 \right) \right\} \left| \varphi^{j,a} \right\rangle =$$
$$\eta_r \frac{1}{2} \left\{ C_{11} \left| j \right\rangle \otimes C_{22} \hat{O}_1 C_{22} \left| \Phi_r^a \right\rangle - C_{21} \left| j \right\rangle \otimes C_{12} \hat{O}_1 C_{22} \left| \Phi_r^a \right\rangle \right.$$
$$\left. - C_{21} \left| j \right\rangle \otimes C_{11} \hat{O}_2 C_{12} \left| \Phi_r^a \right\rangle + C_{11} \left| j \right\rangle \otimes C_{21} \hat{O}_2 C_{12} \left| \Phi_r^a \right\rangle \right\} \qquad (25)$$

If we multiply in (25) from the left with

$$\left\langle j \right| (C_{11} \times \mathbb{1}) \quad \text{or} \quad \left\langle j \right| (C_{12} \times \mathbb{1}) \quad,$$

we get with the normalization of the states $\left| j \right\rangle$ (see 54):

$$\left\{ \omega - m \left(\gamma_1^0 + \gamma_2^0 + \gamma_3^0 \right) \right\} \eta_r C_{22} \left| \Phi_r^a \right\rangle = \frac{1}{2} \left\{ C_{22} \hat{O}_1 + C_{21} \hat{O}_2 C_{12} \right\} \eta_r C_{22} \left| \Phi_r^a \right\rangle \qquad (26)$$

In (26) we have no isospin-superspin dependence i. e. we have completely separated the calculation of the isospin-superspin part from the spin orbit part of the three-particle states. If $\eta_r C_{22} \left| \Phi_r^a \right\rangle$ is given, i. e. is calculated from (26), we get the complete state $\left| \varphi^{j,a} \right\rangle$ with the help of the relation

$$\left| \varphi^{j,a} \right\rangle = \left(\mathbb{1} \times \mathbb{1} - C_{21} \times C_{12} \right) \left\{ C_{11} \left| j \right\rangle \otimes \eta_r C_{22} \left| \Phi_r^a \right\rangle \right\} \qquad (27)$$

where the states $C_{11}|j\rangle$ which fulfill

$$(\gamma_5)_{\kappa_2\kappa_3}\langle\kappa_1\kappa_2\kappa_3|\,C_{11}\,|j\rangle = (\gamma_5)_{\kappa_2\kappa_3}\left(C_{11}\Theta^j\right)_{\kappa_1\kappa_2\kappa_3} = 0 \qquad (28)$$

are given in appendix B.

If projected in configuration space, equation (26) reads:

$$\left\{\omega - m\left((\gamma^0)_{\alpha_1\beta_1}\delta_{\alpha_2\beta_2}\delta_{\alpha_3\beta_3} + (\gamma^0)_{\alpha_2\beta_2}\delta_{\alpha_1\beta_1}\delta_{\alpha_3\beta_3} + (\gamma^0)_{\alpha_3\beta_3}\delta_{\alpha_2\beta_2}\delta_{\alpha_1\beta_1}\right)\right\}\cdot$$

$$\cdot C_{22}(\mathbf{r},\beta)\circ\eta_r\Phi_r\begin{pmatrix}\mathbf{r}_1\ \mathbf{r}_2\ \mathbf{r}_3\\ \beta_1\ \beta_2\ \beta_3\end{pmatrix} = \frac{g}{2}\int d\mathbf{z}_1\,d\mathbf{z}_2\,d\mathbf{z}_3\left\{C_{22}(\mathbf{r},\alpha)\circ\right.$$

$$(\gamma^0\gamma^\mu)_{\alpha_1\beta}\,F\begin{pmatrix}\mathbf{r}_1-\mathbf{r}_2\\ \beta\ \alpha_2\end{pmatrix}(\gamma^\mu C)^\dagger_{\beta_1\beta_2}\delta_{\alpha_3\beta_3}\,\delta(\mathbf{r}_1-\mathbf{z}_1)\,\delta(\mathbf{r}_1-\mathbf{z}_2)\,\delta(\mathbf{r}_1-\mathbf{z}_3)$$

$$+\,C_{21}(\mathbf{r},\alpha)\circ(\gamma^0\gamma^\mu\gamma_5)_{\alpha_1\beta}\,F\begin{pmatrix}\mathbf{r}_1-\mathbf{r}_2\\ \beta\ \alpha_2\end{pmatrix}(\gamma^\mu\gamma_5 C)^\dagger_{\beta_2\beta_1}\delta_{\alpha_3\beta_3}\cdot$$

$$\delta(\mathbf{r}_1-\mathbf{z}_1)\,\delta(\mathbf{r}_1-\mathbf{z}_2)\,\delta(\mathbf{r}_1-\mathbf{z}_3)\,C_{12}(\mathbf{z},\beta)\circ\left.\right\}C_{22}(\mathbf{z},\beta)\circ\eta_r\Phi_r\begin{pmatrix}\mathbf{z}_1\ \mathbf{z}_2\ \mathbf{z}_3\\ \beta_1\ \beta_2\ \beta_3\end{pmatrix}\ (29)$$

The notation $C_{ik}(\mathbf{r},\alpha)$ indicates that the Young-operator C_{ik} act on the indices $\alpha,\mathbf{r}$. Equation (29) can in principle be solved [5]. However, because we are only interested in the structure of the solution rather than in it's detailed form, we only consider (26) for $\mathbf{k}=0$, $\mathbf{r}_1=\mathbf{r}_2=\mathbf{r}_3=\mathbf{r}$ (see also [5]). In this case we can make the ansatz

$$\eta_r C_{22}\Phi_r^a\begin{pmatrix}\mathbf{r}_1\ \mathbf{r}_2\ \mathbf{r}_3\\ \alpha_1\ \alpha_2\ \alpha_3\end{pmatrix}\bigg|_{\substack{\mathbf{k}=0\\ \mathbf{r}_1=\mathbf{r}_2=\mathbf{r}_3}} = \eta_r\,\Omega_r^a(\alpha_1\alpha_2\alpha_3) \qquad (30)$$

Because we are interested in spin-1/2-solutions, we need a complete set of multi-spinors of the third kind in H_{22} which describe spin 1/2-states. Without proof we give the following lemma:

Lemma 3 *The multispinors of the third kind which describe spin 1/2 and which are eigenstates of C_{22} with eigenvalue 1 (i. e. are elements of H_{22}), are unique linear combinations of the following three linear independent multispinors*

$$\hat{\Omega}_1^a = \frac{K_\mu K^\nu}{K^2}\left\{\gamma^\mu C\otimes\gamma_\nu\cdot\hat{\chi}^a(K) + \Sigma^{\mu\rho}C\otimes\Sigma_{\nu\rho}\cdot\hat{\chi}^a(K)\right\}$$
$$\hat{\Omega}_2^a = \gamma^\mu C\otimes\gamma_\mu\cdot\hat{\chi}^a(K)$$
$$\hat{\Omega}_3^a = \Sigma^{\mu\nu}C\otimes\Sigma_{\mu\nu}\cdot\hat{\chi}^a(K)\quad,$$

where we have used the notation $(A\otimes\psi)_{\alpha_1\alpha_2\alpha_3} := A_{\alpha_1\alpha_2}\psi_{\alpha_3}$. Furthermore the Diracspinor $\hat{\chi}^a(K)$ is assumed to fulfill $(\gamma^\mu K_\mu - \sqrt{K^2})\hat{\chi}^a = 0$.

Because we are interested in the case $\mathbf{k}=0$, we have to perform the limit $\mathbf{k}\to 0$ in lemma 3:

$$\hat{\Omega}_1^a\overset{\mathbf{k}=0}{\Longrightarrow}\Omega_1^a = \gamma^0 C\otimes\gamma^0\cdot\chi^a + \Sigma^{0k}\otimes\Sigma_{0k}\cdot\chi^a \qquad (31)$$

10

$$\hat{\Omega}_2^a \overset{\mathbf{k}=0}{\Rightarrow} \Omega_2^a = \gamma^\mu C \otimes \gamma_\mu \cdot \chi^a \tag{32}$$

$$\hat{\Omega}_3^a \overset{\mathbf{k}=0}{\Rightarrow} \Omega_3^a = \Sigma^{\mu\nu} C \otimes \Sigma_{\mu\nu} \cdot \chi^a \tag{33}$$

where

$$\hat{\chi}^a \overset{\mathbf{k}=0}{\Rightarrow} \chi^a \in \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

To solve (26) for the case $\mathbf{r}_1 = \mathbf{r}_2 = \mathbf{r}_3$, $\mathbf{k} = 0$, we substitute the ansatz $\eta_r \Omega_r^a$ into the projected equation of (26). For brevity we do not exhibit the corresponding calculations, rather we give the final result. Combining the η_r into a vector $\vec{\eta}$, we get a homogeneous equation with the structure $A \vec{\eta} = 0$, where the matrix A is given by

$$A := \begin{pmatrix} \omega p + m + 2/3\,\mu & \omega p + m + 2/3\,\mu & 0 \\ 2m + 2/3\,\mu & \omega p - m - 2/3\,\mu & 4m \\ -1/3\,\mu & 1/3\,\mu & \omega p + m \end{pmatrix}$$

$$\mu := g\, s(0) \quad,$$

and $p \in \{1, -1\}$ is the parity of the three-particle states. From $\det A = 0$ we get for the energy eigenvalues ω_i:

$$\omega_1 = -p(m + 2/3\,\mu)$$
$$\omega_2 = p\left[2/3\sqrt{9m^2 + 12m\,\mu + \mu^2} + (m + 2/3\,\mu)\right]$$
$$\omega_3 = -p\left[2/3\sqrt{9m^2 + 12m\,\mu - \mu^2} + (m + 2/3\,\mu)\right]$$

Due to $\mathbf{k} = 0$, the eigenvalues ω_i are the masses of the bound states. They are given as a function of the coupling constant in figure 1, where the region of the coupling constant $g \sim \mu$ has been chosen to yield $|\omega_i| < 3m$. Furthermore, we see that there is a region of g which yields *three* different eigenvalues ω_i, corresponding to *three* linear independent vectors $\vec{\eta}$. But we also recognize that in spite of $|\omega_i| < 3m$, the mass scale of the bound states coincide with the mass scale of the elementary fermions.

Masses of the Boundstates

Figure 1

$\left|\frac{\omega}{m}\right|$

$\mu/m = g\,s(0)/m$

This is unsatisfactory because we assume the masses of the elementary fermions to
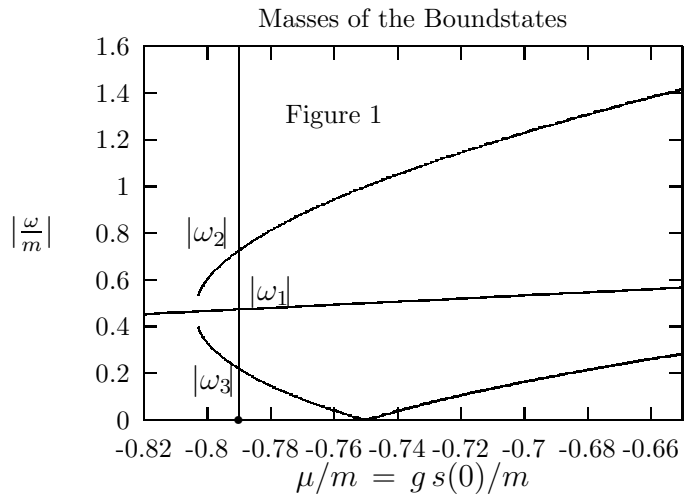
be very high. In order to get realistic masses one is not allowed to use the strong-coupling limit. Furthermore one should use a more realistic propagator instead of the free propagator and in addition one had to take into account the polarization cloud. But we are not interested in numerical values of masses or coupling constants respectively, rather the above discussion should demonstrate the appearance of _three_ solution manifolds which is offered by mixed symmetric spin states.

5 Summary and outlook

In this paper we have calculated a special class of spin $1/2$ solutions of generalized three-particle B. W. equations. The reason why we have concentrated ourselves on solutions with mixed symmetry is the appearance of isospin doublets (see appendix B, table 2) and the appearance of _three_ linearly independent solution manifolds. The calculation of generalized three-particle B. W. equations resulting from the nonlinear spinor equation (1) is only a first step in the calculation of the three-subfermion bound state dynamics. It has already been emphasized in section 3 that the non-diagonal part of (4) mediates the interactions of the three-particle states. It has to be shown in a forthcoming paper that the effective interaction between the mixed symmetric three-particle states and the two-subfermion composites leads to the inclusion of the three generations of leptons and quarks. This effective interaction has to be calculated in the framework of the weak mapping procedure which is a mathematical tool for calculating effective bound state dynamics [2].

Acknowledgement

Appendix

A The Young-operators of S(3) which correspond to mixed Symmetries

In this section we consider those irreducible representation of the group S(3) which correspond to the Young-diagram ⊟. Because we have two possible Young-tableaus (⊞², ⊞³), the representation is two-dimensional and the corresponding four Young-operators C_{ik} are defined as [6]:

$$C_{ik} := \frac{1}{3} \sum_{p \in S(3)} D_{ik}\left(p^{-1}\right) P \tag{34}$$

where D_{ik} are irreducible two-dimensional matrix representations of the permutation group $S(3)$ and P is an operator representation of the abstract $S(3)$-element p. To be definite we choose [7]

$$D(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad\qquad D(p_{12}) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$D(p_{13}) = \begin{pmatrix} 1/2 & -1/2\sqrt{3} \\ -1/2\sqrt{3} & -1/2 \end{pmatrix} \qquad D(p_{23}) = \begin{pmatrix} 1/2 & 1/2\sqrt{3} \\ 1/2\sqrt{3} & -1/2 \end{pmatrix} \qquad (35)$$

$$D(p_{13} \cdot p_{12}) = \begin{pmatrix} -1/2 & -1/2\sqrt{3} \\ 1/2\sqrt{3} & -1/2 \end{pmatrix} \quad D(p_{12} \cdot p_{13}) = \begin{pmatrix} -1/2 & 1/2\sqrt{3} \\ -1/2\sqrt{3} & -1/2 \end{pmatrix}$$

We have denoted the transpositions which interchange j, l by p_{jl}. From (34) and (35) we obtain

$$
\begin{aligned}
C_{11} &= \tfrac{1}{3}\left(2 + P_{13} + P_{23}\right)\tfrac{1}{2}\left(1 - P_{12}\right) = \tfrac{1}{2}\left(1 - P_{12}\right)\tfrac{1}{3}\left(2 + P_{13} + P_{23}\right) \\
C_{22} &= \tfrac{1}{3}\left(2 - P_{13} - P_{23}\right)\tfrac{1}{2}\left(1 + P_{12}\right) = \tfrac{1}{2}\left(1 + P_{12}\right)\tfrac{1}{3}\left(2 - P_{13} - P_{23}\right) \\
C_{12} &= \tfrac{\sqrt{3}}{3}\left(P_{23} - P_{13}\right)\tfrac{1}{2}\left(1 + P_{12}\right) \quad = \tfrac{1}{2}\left(1 - P_{12}\right)\tfrac{\sqrt{3}}{3}\left(P_{23} - P_{13}\right) \\
C_{21} &= \tfrac{\sqrt{3}}{3}\left(P_{23} - P_{13}\right)\tfrac{1}{2}\left(1 - P_{12}\right) \quad = \tfrac{1}{2}\left(1 + P_{12}\right)\tfrac{\sqrt{3}}{3}\left(P_{23} - P_{13}\right)
\end{aligned}
\qquad (36)
$$

where P_{ik} is an operator representation of the S(3)-element p_{ik}. In [6] the following properties of the C_{ik} are proven

$$C^a\,C_{ik} = C^s\,C_{ik} = 0 \qquad\qquad (37)$$
$$C_{ik}^{+} = C_{ki} \qquad\qquad (38)$$
$$C_{ik} \cdot C_{lj} = \delta_{kl}\,C_{ij} \qquad\qquad (39)$$
$$P\,C_{ik} = \left(D^T(p) \cdot C\right)_{ik} \qquad\qquad (40)$$

The matrix D^T is the transposed of the matrix D. In this paper the following applications of (40) are used:

$$P_{23}\,C_{11} = \frac{1}{2}C_{11} + \frac{1}{2}\sqrt{3}C_{21} \qquad\qquad (41)$$

$$P_{13}\,C_{11} = \frac{1}{2}C_{11} - \frac{1}{2}\sqrt{3}C_{21} \qquad\qquad (42)$$

B Isospin States with mixed Symmetry

Due to the global SU(2)×U(1) form invariance of the spinor theory, we can classify the three-particle states $|a\rangle$ according to SU(2) and U(1) quantum numbers, ignoring the possibility of symmetry breaking. Therefore, we require the state $|a\rangle$ to fulfil the equations

$$T^k T^k |a\rangle = t(t+1) |a\rangle, \quad T^3 |a\rangle = t^z |a\rangle, \quad F |a\rangle = f |a\rangle,$$
$$Q |a\rangle = q |a\rangle, \tag{43}$$

where T^k are the generators of the SU(2) transformations which satisfy

$$\left[T^k, \psi_I\right] = -T^k_{II'} \psi_{I'} \quad \text{with} \quad T^k_{II'} := \frac{1}{2} \begin{pmatrix} \sigma^k & 0 \\ 0 & (-)^k \sigma^k \end{pmatrix}_{\kappa\kappa'} \delta_{\alpha\alpha'} \delta(\mathbf{r} - \mathbf{r}') \quad . \tag{44}$$

The U(1) transformations which are generated by F, Q can be directly read off from the relations

$$[F, \psi_I] = -F_{II'} \psi_{I'} \quad \text{with} \quad F_{II'} := \frac{1}{3} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}_{\kappa\kappa'} \delta_{\alpha\alpha'} \delta(\mathbf{r} - \mathbf{r}'), \tag{45}$$

$$[Q, \psi_I] = -Q_{II'} \psi_{I'} \quad \text{with} \quad Q_{II'} := \frac{1}{3} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{\kappa\kappa'} \delta_{\alpha\alpha'} \delta(\mathbf{r} - \mathbf{r}'), \tag{46}$$

$$\tag{47}$$

The SU(2) quantum numbers t, t_z are called isospin quantum numbers whereas the U(1) quantum numbers f, q are called fermion number and charge. Obviously, the transformations generated by T^3, F, Q are not independent. As a consequence, the relation $q = t^z + f/2$ holds.

From (43) we get

$$\frac{9}{4} \varphi_{I_1 I_2 I_3} + 2 \left[T^k_{I_1 I} T^k_{I_2 I'} \varphi_{II'I_3} \right.$$
$$\left. + T^k_{I_1 I} T^k_{I_3 I'} \varphi_{II_2 I'} + T^k_{I_2 I} T^k_{I_3 I'} \varphi_{I_1 II'} \right] = t(t+1) \varphi_{I_1 I_2 I_3} \quad , \tag{48}$$

and:

$$T^3_{I_1 I} \varphi_{II_2 I_3} + T^3_{I_2 I} \varphi_{I_1 II_3} + T^3_{I_3 I} \varphi_{I_1 I_2 I} = t^z \varphi_{I_1 I_2 I_3} \tag{49}$$
$$F_{I_1 I} \varphi_{II_2 I_3} + F_{I_2 I} \varphi_{I_1 II_3} + F_{I_3 I} \varphi_{I_1 I_2 I} = f \varphi_{I_1 I_2 I_3} \tag{50}$$
$$Q_{I_1 I} \varphi_{II_2 I_3} + Q_{I_2 I} \varphi_{I_1 II_3} + Q_{I_3 I} \varphi_{I_1 I_2 I} = q \varphi_{I_1 I_2 I_3} \tag{51}$$

If we substitute the ansatz (17) into (48)-(51), we get equations for the isospin-superspin part $C_{11} |\Theta^j\rangle$, $C_{22} |\Theta^j\rangle$ i. e. we have to discuss isospin states with mixed symmetry, characterized by the Young-diagram ⊞. We restrict ourselves to those

isospin states which correspond to the Young-tableau $\boxed{\begin{smallmatrix}1&3\\2\end{smallmatrix}}$. These states are eigenstates to C_{11} or C_{22} resp. with eigenvalues 1 or 0 resp.. The states which correspond to the tableau $\boxed{\begin{smallmatrix}1&2\\3\end{smallmatrix}}$ can be generated from the $\boxed{\begin{smallmatrix}1&3\\2\end{smallmatrix}}$-states with the help of the operator C_{21}. The number of independent tensors in n dimensions which correspond to a special tableau can be calculated according to [9]. For mixed symmetry we have

$$
\frac{\begin{array}{|c|c|}\hline \text{n} & \text{n+1} \\\hline \text{n-1} \\\cline{1-1}\end{array}}{\begin{array}{|c|c|}\hline 3 & 1 \\\hline 1 \\\cline{1-1}\end{array}} = \frac{(n+1)n(n-1)}{3} \tag{52}
$$

In our case $(n=4)$ we have 20 linear independent isospin states corresponding to $\boxed{\begin{smallmatrix}1&3\\2\end{smallmatrix}}$. For the multiplicity of the states we have

$$
20 = \underbrace{2}_{\text{states + charge conjugated states}} \cdot \left(1 \cdot \underbrace{4}_{\text{quartet}} +3 \cdot \underbrace{2}_{\text{doublet}} \right)
$$

However, we must take into account the condition

$$
(\gamma_5)_{\kappa_2\kappa_3} (C_{11}\Theta^j)_{\kappa_1\kappa_2\kappa_3} = 0 \tag{20}
$$

(see section 4). Because (20) has an open index κ we have four constraints which restrict the number of states to $20 - 4 = 16$. These 16 states can be classified according to SU(2) and U(1) quantum numbers i. e. they can be chosen to fulfill (48)-(51). In order to present these states in a readable form, we introduce the following definitions

$$
T_\kappa^+ := \delta_{1\kappa} \;,\; T_\kappa^- := \delta_{2\kappa} \;,\; V_\kappa^- := \delta_{3\kappa} \;,\; V_\kappa^+ := \delta_{4\kappa} \tag{53}
$$

The U(1), SU(2) quantum numbers of the subfermions are given in the following table:

	T^+	T^-	V^+	V^-
t	1/2	1/2	1/2	1/2
t_z	1/2	-1/2	1/2	-1/2
f	1/3	1/3	-1/3	-1/3
q	2/3	-1/3	1/3	-2/3

Table 1

For simplicity we we omit the index κ. For instance the expression $T^+V^-T^-$ is an abbreviation for $T_{\kappa_1}^+ V_{\kappa_2}^- T_{\kappa_3}^- = \delta_{1\kappa_1}\delta_{3\kappa_2}\delta_{2\kappa_3}$. With these definitions, the 16 isospin states which fulfill (20) are given by

$$
\Theta^1 := \frac{1}{\sqrt{2}} \left(T^+T^-T^+ - T^-T^+T^+ \right)
$$

$$\Theta^2 := \frac{1}{\sqrt{2}}\left(T^+T^-T^- - T^-T^+T^-\right)$$

$$\Theta^3 := \frac{1}{\sqrt{2}}\left(V^+V^-V^+ - V^-V^+V^+\right)$$

$$\Theta^4 := \frac{1}{\sqrt{2}}\left(V^+V^-V^- - V^-V^+V^-\right)$$

$$\Theta^5 := \frac{1}{\sqrt{6}}\left(T^+T^-V^+ - T^-T^+V^+ + V^+T^-T^+ - T^-V^+T^+\right.$$
$$\left. + V^-T^+T^+ - T^+V^-T^+\right)$$

$$\Theta^6 := \frac{1}{\sqrt{6}}\left(T^-T^+V^- - T^+T^-V^- + V^-T^+T^- - T^+V^-T^-\right.$$
$$\left. + V^+T^-T^- - T^-V^+T^-\right)$$

$$\Theta^7 := \frac{1}{\sqrt{6}}\left(V^+V^-T^+ - V^-V^+T^+ + T^+V^-V^+ - V^-T^+V^+\right.$$
$$\left. + T^-V^+V^+ - V^+T^-V^+\right)$$

$$\Theta^8 := \frac{1}{\sqrt{6}}\left(V^-V^+T^- - V^+V^-T^- + T^-V^+V^- - V^+T^-V^-\right.$$
$$\left. + T^+V^-V^- - V^-T^+V^-\right)$$

$$\Theta^9 := \frac{1}{\sqrt{2}}\left(T^+V^+T^+ - V^+T^+T^+\right)$$

$$\Theta^{10} := \frac{1}{\sqrt{6}}\left(T^+V^+T^- - V^+T^+T^- - T^+V^-T^+ + V^-T^+T^+\right.$$
$$\left. + T^-V^+T^+ - V^+T^-T^+\right)$$

$$\Theta^{11} := \frac{1}{\sqrt{6}}\left(T^-V^-T^+ - V^-T^-T^+ - T^-V^+T^- + V^+T^-T^-\right.$$
$$\left. + T^+V^-T^- - V^-T^+T^-\right)$$

$$\Theta^{12} := \frac{1}{\sqrt{2}}\left(T^-V^-T^- - V^-T^-T^-\right)$$

$$\Theta^{13} := \frac{1}{\sqrt{2}}\left(V^+T^+V^+ - T^+V^+V^+\right)$$

$$\Theta^{14} := \frac{1}{\sqrt{6}}\left(V^+T^+V^- - T^+V^+V^- - V^+T^-V^+ + T^-V^+V^+\right.$$
$$\left. + V^-T^+V^+ - T^+V^-V^+\right)$$

$$\Theta^{15} := \frac{1}{\sqrt{6}}\left(V^-T^-V^+ - T^-V^-V^+ - V^-T^+V^- + T^+V^-V^-\right.$$
$$\left. + V^+T^-V^- - T^-V^+V^-\right)$$

$$\Theta^{16} := \frac{1}{\sqrt{2}}\left(V^-T^-V^- - T^-V^-V^-\right)$$

The isospin states $|j\rangle$ which correspond to the matrix elements $\Theta^j \equiv \Theta^j_{\kappa_1\kappa_2\kappa_3} = \langle\kappa_1\kappa_2\kappa_3|\,j\rangle$ fulfill the relation $C_{11}\,|j\rangle = |j\rangle$ and are normalized according to

$$\langle j'|\,C_{11}\,|j\rangle = \langle j'|\,j\rangle = \delta_{jj'} \quad . \tag{54}$$

The verification of (20) is best done with the representation

$$\gamma_5 = T^+V^- + V^-T^+ + V^+T^- + T^-V^+$$

The quantum numbers of the above isospin states are summarized in the following table

$|j\rangle$	$|1\rangle$	$|2\rangle$	$|3\rangle$	$|4\rangle$	$|5\rangle$	$|6\rangle$	$|7\rangle$	$|8\rangle$
t	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2
t^z	1/2	-1/2	1/2	-1/2	1/2	-1/2	1/2	-1/2
f	1	1	-1	-1	1/3	1/3	-1/3	-1/3
q	1	0	0	-1	2/3	-1/3	1/3	-2/3
Interpretation:								
ω_3	e^+	$\bar{\nu}_e$	ν_e	e^-	u	d	$\bar{d}$	$\bar{u}$
ω_1	μ^+	$\bar{\nu}_\mu$	ν_μ	μ^-	c	s	$\bar{s}$	$\bar{c}$
ω_2	τ^+	$\bar{\nu}_\tau$	ν_τ	τ^-	t	b	$\bar{b}$	$\bar{t}$

Table 2

$|j\rangle$	$|9\rangle$	$|10\rangle$	$|11\rangle$	$|12\rangle$	$|13\rangle$	$|14\rangle$	$|15\rangle$	$|16\rangle$
t	3/2	3/2	3/2	3/2	3/2	3/2	3/2	3/2
t^z	3/2	1/2	-1/2	-3/2	3/2	1/2	-1/2	-3/2
f	1/3	1/3	1/3	1/3	-1/3	-1/3	-1/3	-1/3
q	5/3	2/3	-1/3	-4/3	4/3	1/3	-2/3	-5/3

At this stage of calculation the name isospin, fermion number and charge do not imply any physical interpretation of the states. Rather these quantum numbers reflect the symmetry of eqn. (6) and serve as bookkeeping indices only. In order to determine the phenomenological quantum numbers of the three-subfermion bound states, the interaction with other bound state particles must be taken into account. However, anticipating the results of a forthcoming paper in which the effective interaction of these three-particle states with the two-fermion composites will be calculated, we may identify the quantum numbers t, t^z, f, q as well as the hypercharge $y = t^z + \frac{f}{2}$ with the phenomenological quantum numbers of leptons and quarks.

References

[1] W. Pfister, M. Rosa and H. Stumpf, Nuovo Cim. **102 A** (1989) 1449.

[2] H. Stumpf and Th. Borne, Composite Particle Dynamics in Quantum Field Theory, Vieweg Verlag, Wiesbaden/Braunschweig 1994

[3] H. Stumpf, Z. Naturforsch. **41a**, 683 (1986); **41a**, 1399 (1986); **42a**, 213 (1987)

[4] H. Stumpf, Z. Naturforsch. **37a** (1982) 1295;
D. Grosser, Z. Naturforsch. **38a** (1983) 1293.

[5] W. Pfister, Nuovo Cim. **107A**, 1523 (1994)

[6] P. Kramer, G. John and D. Schenzle, Group Theory and the Interaction of Composite Nucleon systems, Vieweg Verlag, Wiesbaden/Braunschweig 1981

[7] T. Lauxmann, thesis, University of Tübingen, (1986).

[8] W. Pfister and H. Stumpf, accepted for publication in Nuovo Cim.

[9] D. E. Littlewood, The Theory of Group Characters (Clarendon Press, Oxford 1950)